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FIELDS AND SYMMETRIES FROM CY COMPACTIFICATION

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FIELDS AND SYMMETRIES FROM CY COMPACTIFICATION I

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Complex (Analytic) Manifolds

Def A Hausdorff topological space M with an open covering U_i $(i \in I)$ $\left[\bigcup_{i \in I} U_i = M \,,\ U_i = \text{open sets} \right]$ and homeomorphisms φ_i of U_i's onto open sets of $\mathbb{C}^n$ such that if $U_i \cap U_j \neq \phi$

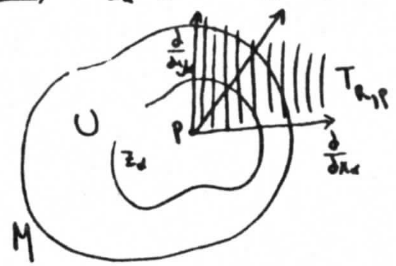
$\varphi_i \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \longrightarrow \varphi_i(U_i \cap U_j)$ is biholomorphic

$z^{(i)}_\alpha(z^{(j)}_\beta)$: holomorphic fns

$\alpha, \beta = 1, \dots, n$

$i, j \in I$

$\longrightarrow$ $z_\alpha = x_\alpha + i y_\alpha$ $\rightarrow$ (x_α, y_α) are $2n$ real coordinates and M becomes $2n$-dim. real C^∞-manifold.



Real tangent space : $T_{R,p}$ is generated by $\left(\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial y_\alpha}\right)$

$a_\alpha \frac{\partial}{\partial x_\alpha} + b_\alpha \frac{\partial}{\partial y_\alpha} \in T_{R,p}$ $\forall\, a_\alpha, b_\alpha \in \mathbb{R}$ (2n real dim.)

Complexified tangent space $T_{\mathbb{C},z}$: $a_\alpha \frac{\partial}{\partial x_\alpha} + b_\alpha \frac{\partial}{\partial y_\alpha} \in T_{\mathbb{C},p}$, $\forall\, a_\alpha, b_\alpha \in \mathbb{C}$

(2n complex dim = 4n real dim.)

Now since $\frac{\partial}{\partial z_\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x_\alpha} - i \frac{\partial}{\partial y_\alpha}\right)$, $\frac{\partial}{\partial \bar z_\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x_\alpha} + i \frac{\partial}{\partial y_\alpha}\right)$ or inversely

$$\frac{\partial}{\partial x_\alpha} = \frac{\partial}{\partial z_\alpha} + \frac{\partial}{\partial \bar z_\alpha} \,, \quad \frac{\partial}{\partial y_\alpha} = i\left(\frac{\partial}{\partial z_\alpha} - \frac{\partial}{\partial \bar z_\alpha}\right)$$

any vector in $T_{\mathbb{C},z}$ can be written as

$$\underbrace{\lambda_\alpha \frac{\partial}{\partial z_\alpha}}_{\text{holomorphic}} + \underbrace{\nu_\alpha \frac{\partial}{\partial \bar z_\alpha}}_{\text{antiholomorphic}} \,, \quad \lambda_\alpha, \nu_\alpha \in \mathbb{C}$$

So $T_{\mathbb{C},z} = T'_p \oplus T''_p$ (this decomposition is coordinate independent , since $z^{(i)}_\alpha = z^{(i)}_\alpha(z^{(j)}_\beta) \rightarrow \frac{\partial}{\partial z^{(j)}_\beta} = \frac{\partial z^{(i)}_\alpha}{\partial z^{(j)}_\beta} \frac{\partial}{\partial z^{(i)}_\alpha}$)

conjugation : $\lambda_\alpha \frac{\partial}{\partial z_\alpha} \rightarrow \bar\lambda_\alpha \frac{\partial}{\partial \bar z_\alpha}$; $T' \rightarrow T''$

Holomorphic vector fields $\lambda_\alpha(z) \frac{\partial}{\partial z_\alpha}$

Anti- " " " $\nu_\alpha(\bar z) \frac{\partial}{\partial \bar z_\alpha}$

Similarly the real cotangent space $T^*_{R,p}$ generated by (dx_α, dy_α) can be complexified

$$T^*_{\mathbb{C},p} \ni a^\alpha dx_\alpha + b^\alpha dy_\alpha \qquad a^\alpha, b^\alpha \in \mathbb{C}$$

$$T^{*\prime}_p \oplus T^{*\prime\prime}_p = \underbrace{\lambda^\alpha dz_\alpha}_{\text{holomorphic}} + \underbrace{\nu^\alpha d\bar z_\alpha}_{\text{antiholomorphic}} \qquad \lambda^\alpha, \nu^\alpha \in \mathbb{C}$$

$(dz_\alpha = dx_\alpha + i dy_\alpha ,\ d\bar z_\alpha = dx_\alpha - i dy_\alpha)$

Holomorphic 1-form $\lambda^\alpha(z)\, dz_\alpha$

Anti " " $\nu^\alpha(\bar z)\, d\bar z_\alpha$

$A^{p,q}(M) = \Lambda^p T^{*\prime}(M) \wedge \Lambda^q T^{*\prime\prime}(M)$ $\ni \omega = \omega^{\alpha_1 \dots \alpha_p \beta_1 \dots \beta_q}(z,\bar z)\, dz_{\alpha_1} \wedge \dots \wedge dz_{\alpha_p} \wedge d\bar z_{\beta_1} \wedge \dots \wedge d\bar z_{\beta_q}$

(p,q) - forms

p : holomorphic , q : antiholomorphic

$$\boxed{A^r(M) = \bigoplus_{p+q=r} A^{p,q}(M)}$$

A^r : r-forms

exterior differential:

$$d:\ \omega \to d\omega = \frac{\partial \omega^{i_1\cdots i_p j_1\cdots j_q}(z,\bar z)}{\partial z_i}\, dz_i\wedge dz_{i_1}\wedge\cdots\wedge dz_{i_p}\wedge d\bar z_{j_1}\wedge\cdots\wedge d\bar z_{j_q}$$

$$+(-)^p\frac{\partial \omega^{i_1\cdots i_p j_1\cdots j_q}}{\partial \bar z_j}\, dz_{i_1}\wedge\cdots\wedge dz_{i_p}\wedge d\bar z_j\wedge d\bar z_{j_1}\wedge\cdots\wedge d\bar z_{j_q}$$

holomorphic $\quad \equiv \partial\omega + \bar\partial\omega$

$\to \underline{d = \partial + \bar\partial}$ antiholomorphic

d : r-forms $\to$ $(r+1)$-forms
∂ : (p,q)-" $\to$ $(p+1,q)$-"
$\bar\partial$: (p,q)-" $\to$ $(p,q+1)$-forms

<u>Poincaré lemma</u>: $\partial^2 = \bar\partial^2 = 0 \quad (d^2=0)$

for example $$\bar\partial^2\omega = \frac{\partial^2\omega^{i_1\cdots i_p j_1\cdots j_q}(z\bar z)}{\partial\bar z_i\,\partial\bar z_j}\, d\bar z_i\wedge d\bar z_j\wedge dz_{i_1}\wedge\cdots\wedge dz_{i_p}\wedge d\bar z_{j_1}\wedge\cdots\wedge d\bar z_{j_q}$$

$$= 0$$

We can now define cohomologies of (p,q)-forms with respect to $\bar\partial$

$\bar\partial\omega = 0$: closed form
$\omega = \bar\partial f$: exact form
f: $(p,q-1)$-form

Poincaré lemma $\to$ exact forms are always closed.

<u>Dolbeault cohomology</u>: $H^{p,q}_{\bar\partial}(M)$ = the space of all $\bar\partial$-closed (p,q)-forms with two of them identified if they differ by a $\bar\partial$-exact form
= complex linear spaces of finite dim.

Hodge numbers: $h^{p,q}(M) = \dim H^{p,q}(M)$ = complex dim.

V.A.

<u>Hodge diamond</u>

$$\begin{array}{ccccccc}
 & & & h^{0,0} & & & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & h^{2,0} & & h^{1,1} & & h^{0,2} & \\
 & \vdots & & & & \vdots & \\
h^{n,0} & h^{n-1,1} & & \cdots & & h^{1,n-1} & h^{0,n} \\
 & \vdots & & & & \vdots & \\
 & h^{n-2,n} & & h^{n-1,n-1} & & h^{n,n-2} & \\
 & & h^{n-1,n} & & h^{n,n-1} & & \\
 & & & h^{n,n} & & &
\end{array}$$

H.A.

Properties: i) $h^{p,q}(M) = h^{n-p,n-q}(M)$ (Poincaré duality)
$\to$ Hodge diamond is symmetric about the horizontal axis.

ii) $h^{n,n}(M) = 1$ (const. fns) $\qquad$ i)+ii) $\to h^{0,0} = 1$ (const. fns)

<u>Kähler manifolds</u>:

Hermitian metric: $g = g^{\alpha\beta}\, dz_\alpha\, d\bar z_\beta$
$g^{\alpha\beta}$ = positive definite hermitian matrix

Kähler metric

Kähler form : $\Omega = \frac{i}{2} g^{\alpha\beta}\, dz_\alpha\wedge d\bar z_\beta$ $\qquad$ $(1,1)$-form

Kähler manifold: $d\Omega = 0$ (the Kähler form is closed)

$\to \quad g^{\alpha\beta} = \frac{\partial^2\psi}{\partial z_\alpha\,\partial\bar z_\beta}$, ψ = Kähler potential

<u>All manifolds we will deal with are Kähler ($\mathbb{CP}^n$ and Algebraic Varieties in it are Kähler)</u>

Extra properties of Hodge number for Kähler manifolds

α) $h^{p,q}(M) = h^{q,p}(M)$ (symmetry about vertical axis of Hodge diamond)

β) $\sum_{p+q=r} h^{p,q}(M) = b_r(M) = \dim H^r(M) =$ Betti #

$\longrightarrow \chi(M) = \sum_{r=0}^{n} (-1)^r b_r(M) = \sum_{p,q=0}^{n} (-1)^{p+q} h^{p,q}(M)$

Euler characteristic

Calabi-Yau spaces are Kähler manifolds with complex dim. 3 and first Chern class $c_1(M)=0 \longrightarrow$ a) Ricci flat $(R_{ij}=0)$ b) SU(3) holonomy } w.r.t. Kähler metric

(A) The holomorphic tangent vectors $a_\alpha \frac{\partial}{\partial z_\alpha} \in T'$ and the (0,1)-forms $\nu^\alpha d\bar{z}_\alpha$ transform like ~~antitriplets~~ $\bar{3}$ under the SU(3) holonomy.

(B) The antiholomorphic tangent vectors $b_\alpha \frac{\partial}{\partial \bar{z}_\alpha}$ and the (1,0)-forms $\lambda^\alpha dz_\alpha \in \Omega'$ transform like triplets 3 under the SU(3) holonomy.

Also for CY-spaces (a) $h^{1,0} = h^{2,0} = 0 \longrightarrow h^{0,1} = h^{0,2} = h^{2,3} = h^{1,3} = h^{3,2} = h^{3,1} = 0$

(b) $h^{3,0} = h^{0,3} = 1 \longleftrightarrow$ there is one non-vanishing covariantly const. (3,0)-form $\epsilon = \epsilon^{\alpha\beta\gamma} dz^\alpha \wedge dz^\beta \wedge dz^\gamma$

$\Longrightarrow a_\alpha \frac{\partial}{\partial z_\alpha} \longleftrightarrow a_\alpha \epsilon^{\alpha\beta\gamma} dz^\beta \wedge dz^\gamma$

holomorphic tangent bundle $= T' \overset{\sim}{\longleftrightarrow} \Omega^2 =$ the bundle of holomorphic (2,0)-forms

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Spinors on CY-manifolds

The Dirac operator on a CY-manifold $\not D = \gamma_i D^i = \sum_{i=1}^{3} a_\alpha \frac{D}{Dz_\alpha} + \sum_{\alpha=1}^{3} a^*_\alpha \frac{D}{D\bar{z}_\alpha}$. Here $a^{*\alpha} = g^{\beta\alpha} a_\beta$, $a^\alpha = g^{\alpha\beta} a^*_\beta$ $(\alpha = 1,2,3)$ are annihilation and creation operators with $\{a^\alpha, a^\beta\} = \{a^{*\alpha}, a^{*\beta}\} = 0$, $\{a^\alpha, a^{*\beta}\} = g^{\alpha\beta}$

(0,1)-form

The spinor states are: $|0\rangle =$ vacuum (0,0)-form

(0,1)-form $\rightarrow a^{*\alpha}|0\rangle$

(0,2) $\rightarrow a^{*\alpha} a^{*\beta}|0\rangle$

(0,3) $\rightarrow a^{*\alpha} a^{*\beta} a^{*\gamma}|0\rangle$

positive chirality: 4 of SU(4) = Spin(6)

$\bar{4}$ of SU(4) = Spin(6)! negative chirality (since multiplication by a or a^* changes chirality).

Fermi statistics $\rightarrow$ complete antisymmetry in all indices

The first and the last are singlets under SU(3) holonomy $a^{*\alpha}|0\rangle$ is $\bar{3}$ and $a^{*\alpha} a^{*\beta}|0\rangle$ is 3.

* Thus, R-handed spinors are (0,0) and (0,2)-forms
L- " " " (0,1) " (0,3)- "

In discussing the families of E_6 $27, \overline{27}$'s of spinors we observe that, since

Adjoint of $E_8 \rightarrow \underline{248} = (8,1) + (3,27) + (\bar{3},\overline{27}) + (1,78)$

under $E_8 \supset SU(3) \times E_6$

So the 27's ($\overline{27}$'s) of E_6 are 3's ($\bar{3}$'s) of SU(3). But this SU(3) subgroup of E_8 is identified with the SU(3) holonomy of the CY-manifold. (We will assume the identification $SU(3) \rightarrow SU(3)^{-1}$ so that 3's $\rightarrow$ $\bar{3}$'s. This is just a matter of convenient definition)

All this means that our spinors carry extra indices. The 27's being $\bar{3}$'s of the SU(3) holonomy carry an extra holomorphic tangent vector index. By using the $\epsilon^{\alpha\beta\gamma}$, this index is transformed into two holomorphic form indices. So the $\underline{27}$ spinors are (2,q)-form

The $\overline{27}$'s being 3's of SU(3) holonomy carry an extra holomorphic form index.

So the $\overline{27}$'s are $(1,q)$-forms

Combining this observation with the previous discussion of spinors on CY-space we obtain

R-handed 27's	are	$(2,0)$ and $(2,2)$ - forms	
L-handed 27's	"	$(2,1)$ " $(2,3)$ - "	
R- " $\overline{27}$'s	"	$(1,0)$ " $(1,2)$ - "	
L- " $\overline{27}$'s	"	$(1,1)$ " $(1,3)$ - "	

Zero-modes of the Dirac operator

One can prove that zero-modes of the Dirac operator ($\not{D}\psi=0$) on a CY-space are equivalent to forms of type (p,q) with $d\psi=d^*\psi=0$

These are the harmonic forms of type (p,q) $\longrightarrow$ $\mathcal{H}^{p,q}(M)$

Definition of d^*

The Kähler form Ω can be written as

$$\Omega = \frac{i}{2}\,\varphi_j \wedge \bar{\varphi}_j \qquad (g = \varphi_j \otimes \bar{\varphi}_j)$$

where $(\varphi_1, \varphi_2, \cdots \varphi_n)$ are orthonormal holomorphic 1-forms.

In the space of (p,q)-forms we can a hermitian inner product by taking the basis $\varphi_{i_1}\wedge\cdots\wedge\varphi_{i_p}\wedge\bar{\varphi}_{j_1}\wedge\cdots\wedge\bar{\varphi}_{j_q}$ to be orthogonal and of length square 2^{p+q} (Remember $\|dz\|^2=2$). So for two such forms ψ,η we define a complex function

$$(\psi(z),\eta(z)) = \psi^{i_1\cdots i_p j_1\cdots j_q}\,\overline{\eta^{i_1\cdots i_p j_1\cdots j_q}}\; 2^{p+q}$$

if $\psi = \psi^{i_1\cdots i_p j_1\cdots j_q}\,\varphi_{i_1}\wedge\cdots\wedge\bar{\varphi}_{j_q}$, $\eta = \cdots$

we also define the volume form $V = \dfrac{\Omega^n}{n!} = (-1)^{\frac{n(n-1)}{2}}\left(\dfrac{i}{2}\right)^n \varphi_1\wedge\cdots\wedge\varphi_n\wedge\bar{\varphi}_1\wedge\cdots\wedge\bar{\varphi}_n$

$\longrightarrow$ Global inner product

$$(\psi,\eta) = \int_M (\psi(z),\eta(z))\, V$$

linear combinations of various (p,q)-forms

(This product can be extended to any forms by obvious generalization)

$\bar{\partial}^*$: (p,q)-forms $\longrightarrow$ $(p,q-1)$-forms

$$(\bar{\partial}^*\psi,\eta) = (\psi,\bar{\partial}\eta)$$

Similarly $\partial^* \longrightarrow \quad d^* = \partial^* + \bar{\partial}^*$

The Laplace-Beltrami operator $\Delta_d = dd^* + d^*d$ for CY-manifolds

$$\Delta_d = 2(i\not{D})^2$$

$i\not{D}\psi = 0 \;\rightarrow\; (\psi,(i\not{D})^2\psi)=0 \;\rightarrow\; (\psi,(dd^*+d^*d)\psi)=0$

$\rightarrow (\psi,dd^*\psi) + (\psi,d^*d\psi) = 0 \rightarrow (d^*\psi,d^*\psi)+(d\psi,d\psi)=0$

$\rightarrow d\psi = d^*\psi = 0$

So the zero-modes can be represented by harmonic forms of type (p,q) $\rightarrow \mathcal{H}^{p,q}(M)$

$\mathcal{H}^{p,q}(M)$ does not change if we replace $d \rightarrow \bar{\partial}$ or ∂ in the definition ($\bar{\partial}\psi=0$, $\bar{\partial}^*\psi=0$), since for Kähler manifolds $\Delta_d = 2\Delta_{\bar{\partial}} = 2\Delta_{\partial}$

Dolbeault theorem : $\mathcal{H}^{p,q}(M) \simeq H^{p,q}_{\bar{\partial}}(M)$

So we get the following zero-modes

$h^{1,1}$	R-handed	27's	$(h^{2,2}= h^{1,1},\ h^{2,0}=0)$
$h^{2,1}$	L - "	27's	$(h^{2,3} = h^{1,0} = 0)$
$h^{2,1}$	R - "	$\overline{27}$'s	$(h^{1,2}= h^{2,1},\ h^{1,0}=0)$
$h^{1,1}$	L - "	$\overline{27}$'s	$(h^{1,3}= h^{2,0}=0)$

antiparticles

Keeping only L-handed fields :

i) $h^{2,1}$ 27's

ii) $h^{1,1}$ $\overline{27}$'s

$$\chi(M) = \sum_{p,q=0}^{3} (-1)^{p+q}\, h^{p,q} = 2(h^{1,1}-h^{2,1})$$

$$\#\text{ of generations} = h^{2,1}-h^{1,1} = \frac{-\chi(M)}{2}$$

A CY-space with 3 generations

CP^n : all $(x_0, x_1, x_2, \cdots, x_n)$ (homogeneous coordinates) with $x_i \in \mathbb{C}$ except $(0,0,\cdots,0)$
and with the identification of $(x_0 x_1 \cdots x_n) \sim \lambda (x_0 x_1 \cdots x_n)$
$0 \neq \lambda \in \mathbb{C}$

CP^n is n-dim. complex analytic manifold: open covering
U_i $(i=0,1,\cdots,n)$ $[U_i = \text{all } (x_0 x_1 \cdots x_n) \text{ with } x_i \neq 0]$. Coordinates
on U_i : $z_1 = x_0/x_i$, $z_2 = x_1/x_i$, $\cdots$ $z_i = x_{i-1}/x_i$, $z_{i+1} = x_{i+1}/x_i$
$\cdots$ $z_n = x_n/x_i$.

CP^n is Kähler: $\psi_i = \frac{1}{2\pi} \ln (1 + \sum_j |z_j|^2)$ = Kähler
potential in U_i $\longrightarrow$ $g^{\alpha\beta} = \frac{\partial \psi}{\partial z_\alpha \partial \bar{z}_\beta}$

Algebraic Varieties (submanifolds defined by algebraic equations)
in CP^n or $CP^n \times CP^m$, .. are also Kähler
(Any compact complex manifold that can be embedded in CP^n
is Kähler)

Definition of the CY-space: We take $CP^3 \times CP^3$ with homogeneous coordinates $(x_0 x_1 x_2 x_3)$ and $(y_0 y_1 y_2 y_3)$ respectively and impose 3 homog. (in each) eqs. of bidegree $(3,0)$, $(0,3)$, $(1,1)$:

$$\sum_{i,j,k=0}^{3} a_{ijk} x_i x_j x_k = 0, \quad \sum_{i,j,k=0}^{3} b_{ijk} y_i y_j y_k = 0 \qquad \sum_{i,j=0}^{3} c_{ij} x_i y_j = 0$$

The resulting space R_0 is a 3-dim complex manifold which is CY-space. $(c_1 = 0)$

Also, $\chi(R_0) = -2nm(4-n)(4-m) = -18$ for bidegrees $(n,0)$ $(0,m)$ $(1,1)$

Due to symmetry and redundance of an over all factor the independent a_{ijk}'s are $\frac{4.5.6}{3!} - 1 = 19$. The $GL(4)$ has 16 independent parameters but an overall rescaling is irrelevant in our case. So by general linear transformations among the x's we can eliminate 15 out of the 19 parameters. One can show that the cubic can be given the form

$$\sum_{i=0}^{3} x_i^3 + a_1 x_0 x_1 x_2 + a_2 x_0 x_1 x_3 + a_3 x_0 x_2 x_3 + a_4 x_1 x_2 x_3 = 0 \quad (4 \text{ param})$$

Also the cubic in y's becomes

$$\sum_{i=0}^{3} y_i^3 + b_1 y_0 y_1 y_2 + b_2 y_0 y_1 y_3 + b_3 y_0 y_2 y_3 + b_4 y_1 y_2 y_3 = 0 \quad (4 \text{ param})$$

In the polynomial of bidegree $(1,1)$ we can eliminate one parameter $(c_{00} = 1)$:

$$x_0 y_0 + \sum_{\substack{(i,j) \neq (0,0) \\ i,j=0}}^{3} c_{ij} x_i y_j = 0 \quad (15 \text{ param.})$$

Kodaira's deformation theory:

The space of complex structure deformations of a compact complex manifold M is the cohomology group $H^1(M, T')$ of holomorphic one forms with values in the holomorphic tangent vector bundle T' of M. ($\omega = \omega^\alpha{}_\beta(z)\, dz_\alpha$) holomorphic vector index

For M an algebraic variety the linearly independent deformations can be represented by the linearly independent homogeneous monomials that can be used to define M.

For CY-spaces $T' \simeq \Omega^{2} \rightarrow H^1(M, T') = H^1(M, \Omega^2)$

$= H_{\bar{\partial}}^{2,1}(M)$ (Dolbeault theorem)

So the linearly independent deformations are $h^{2,1}(R_0) = 4+4+15 = 23$ and are represented by the monomials (which then represent indep. elem. of $H_{\bar{\partial}}^{2,1}(M)$):

$x_0x_1x_2\ ,\ x_0x_1x_3,\ x_0x_2x_3\ ,x_1x_2x_3,\ y_0y_1y_2, y_0y_1y_3, y_0y_2y_3, y_1y_2y_3,\ x_iy_j\ ((i,j)\neq(0,0))$

So we have 23 $\underline{27}$'s which are represented by the above monomials

From $\chi(R_0) = -18 = 2(h^{1,1} - h^{2,1}) \longrightarrow h^{1,1} = 14$ ← we can derive this independently (see below)

we also have 14 $\overline{\underline{27}}$'s (which are not represented by monomials)

$$\longrightarrow \quad \text{Hodge diamond}(R_0) = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 14 & & 0 & \\ 1 & & 23 & & 23 & & 1 \\ & 0 & & 14 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

To reduce the # of families to 3 and also to be able to apply flux breaking we divide the simply connected manifold R_0 by a discrete group $G \simeq Z_3$ of transformations of R_0 which acts freely on R_0 (no fixed points).

G is generated by
$$g: (x_0x_1x_2x_3) \longrightarrow (x_0, \alpha^2 x_1, \alpha x_2, \alpha x_3)$$
$$(y_0y_1y_2y_3) \longrightarrow (y_0, \alpha y_1, \alpha^2 y_2, \alpha^2 y_3)$$
$$\alpha = \exp\left(\frac{2\pi i}{3}\right)$$

For G to act within R_0 we must restrict the defining polynomials to contain only monomials invariant under G:

$$\sum_{i=0}^{3} x_i^3 + a_1 x_0x_1x_2 + a_2 x_0x_1x_3 = 0$$
$$\sum_{i=0}^{3} y_i^3 + b_1 y_0y_1y_2 + b_2 y_0y_1y_3 = 0$$
$$x_0y_0 + c_1 x_1y_1 + c_2 x_2y_2 + c_3 x_3y_3 + c_4 x_2y_3 + c_5 x_3y_2 = 0$$

[proof that g acts freely on R_0: $(x_0, \alpha^2 x_1, \alpha x_2, \alpha x_3) = \lambda(x_0x_1x_2x_3)$

for a fixed point $\longrightarrow$ $x_0 = x_1 = 0$ and x_2x_3 arbitrary. Similarly, $y_0 = y_1 = 0$

From the cubics $\longrightarrow$ $x_2 \propto x_3$, $y_2 \propto y_3$. From $(1,1) \longrightarrow x_3y_3 = 0$

$\longrightarrow$ either $x_3 = 0$ or $y_3 = 0$. But $x_3 = 0 \longrightarrow x_2 = 0$ which is not allowed]

The Euler characteristic of $R \equiv R_0/G$ is

$$\chi(R) = \frac{-18}{3} = -6 \longrightarrow \text{3 families}$$

From the G invariant independent monomials $\longrightarrow h^{2,1}(R) = 2+2+5 = 9$

$\longrightarrow h^{1,1} = 6$

$$\text{Hodge diamond}(R) = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 6 & & 0 & \\ 1 & & 9 & & 9 & & 1 \\ & 0 & & 6 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

To achieve flux breaking we map G into a Z_3 subgroup of E_6:

$$g \longrightarrow U_g \qquad (\ \psi(gx) = U_g \psi(x)\ )$$

$$E_6 \supseteq SU(3)_c \times SU(3)_L \times SU(3)_R$$

$$U_g \equiv (\alpha \mathbb{1}_3, \mathbb{1}_3, \mathbb{1}_3) = (\mathbb{1}_3, \alpha\mathbb{1}_3, \alpha\mathbb{1}_3)$$

(this last equality can be understood by noting that both group elements act the same way on $\underline{27} = (1,3,\bar{3}) + (\bar{3},1,\bar{3}) + (3,3,1)$)

This breaks $E_6 \longrightarrow SU(3)_c \times SU(3)_L \times SU(3)_R$

Under U_g the leptons $\lambda = (1,3,\bar{3})$, quarks $q = (3,3,1)$ and antiquarks $Q = (\bar{3},1,\bar{3})$ transform like

$$U_g: \quad \lambda \longrightarrow \lambda$$
$$q \longrightarrow \alpha q$$
$$Q \longrightarrow \alpha^2 Q \qquad \text{(all L-handed fields)}$$

Since g is identified with U_g, and we know how g acts on the monomials which represent the 23 (2,1) harmonic forms we can classify these monomials as leptons, quarks, antiquarks

$\lambda_1 = x_0 x_1 x_2$	$q_1 = x_1 x_2 x_3$	$Q_1 = x_0 x_2 x_3$
$\lambda_2 = x_0 x_1 x_3$	$q_2 = y_0 y_2 y_3$	$Q_2 = y_1 y_2 y_3$
$\lambda_3 = y_0 y_1 y_2$	$q_3 = x_0 y_1$	$Q_3 = x_1 y_0$
$\lambda_4 = y_0 y_1 y_3$	$q_4 = x_1 y_2$	$Q_4 = x_0 y_2$
$\lambda_5 = x_1 y_1$	$q_5 = x_1 y_3$	$Q_5 = x_0 y_3$
$\lambda_6 = x_2 y_2 + x_3 y_3$	$q_6 = x_2 y_0$	$Q_6 = x_2 y_1$
$\lambda_7 = x_2 y_2 - x_3 y_3$	$q_7 = x_3 y_0$	$Q_7 = x_3 y_1$
$\lambda_8 = x_2 y_3$		
$\lambda_9 = x_3 y_2$		

So we have 9 leptons + 7 quarks + 7 antiquarks = 23

To study the harmonic (1,1) forms which correspond to $\overline{27}$ zero-modes (they are not represented by monomials and Kodaira's theory is not applicable) we need the Lefschetz fixed point theorem:

Suppose $f: M \to M$ is an isometry of an n-dim Kähler manifold

$\longrightarrow$ Lefschetz number : $L(f) \equiv \sum_{p,q=0}^{n} (-1)^{p+q} \, \mathrm{Tr}_{H^{p,q}_{\bar\partial}(M)}(f)$

[$f: M \to M$ induces a map $f: H^{p,q}_{\bar\partial}(M) \to H^{p,q}_{\bar\partial}(M)$; $\mathrm{Tr}_{H^{p,q}_{\bar\partial}(M)}(f)$ is its trace]

$\longrightarrow$ $L(\text{identity}) = \sum_{p,q=0}^{n} (-1)^{p+q} h^{p,q}(M) = \chi(M)$

Lefschetz fixed point theorem : $L(f) = \sum_{M_j} \chi(M_j)$, where M_j are the submanifold of M that are left fixed by f.

The Hodge diamond of a cubic Σ (2-dim.) in $\mathbb{C}P^3$ is

$$\text{Hodge diamond}(\Sigma) = \begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 7 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix}$$

$\longrightarrow$ $h^{1,1}(\Sigma) = 7$ and since $h^{1,1}(R_0) = 14$ $\longrightarrow$

$$H^{1,1}_{\bar\partial}(R_0) = H^{1,1}(\Sigma_1) \oplus H^{1,1}(\Sigma_2),$$

where Σ_1, Σ_2 the cubics in the first and second $\mathbb{C}P^3$

Lefschetz hyperplane theorem : The above forms are "independent" of the bidegree (1,1) polynomial (hyperplane).

Define the extensions of g to Σ_1 and Σ_2

$$g_1 : (x_0 x_1 x_2 x_3) \longrightarrow (x_0, \alpha^2 x_1, \alpha x_2, \alpha x_3)$$
$$g_2 : (y_0 y_1 y_2 y_3) \longrightarrow (y_0, \alpha y_1, \alpha^2 y_2, \alpha^2 y_3)$$

Lefschetz hyperplane theorem $\longrightarrow \mathrm{Tr}_{H^{1,1}(R_0)}(g) = \mathrm{Tr}_{H^{1,1}(\Sigma_1)}(g_1) +$

$+ \mathrm{Tr}_{H^{1,1}(\Sigma_2)}(g_2)$

To calculate $\mathrm{Tr}_{H^{1,1}(\Sigma_1)}(g_1)$ we employ the Lefschetz fixed point theorem

$$L(g_1) = \sum_{\mu_i} \chi(\mu_j) = 2 + \mathrm{Tr}_{H^{1,1}(\Sigma_1)}(g_1) .$$

Here we took into account the Hodge diamond of Σ_1 and the fact that the $(0,0)$ and $(2,2)$-forms are invariant under g_1 since they are const functions (because $H^{0,0}$ and $H^{2,2}$ are 1-dim.) $\to \mathrm{Tr}_{H^{0,0}(\Sigma)}(g_1) = 1$, $\mathrm{Tr}_{H^{2,2}(\Sigma)}(g) = 1$.

To find the fixed point of g_1 on Σ_1 we take (for a fixed point)

$$g_1 : (x_0 x_1 x_2 x_3) \to (x_0, \alpha^2 x_1, \alpha x_2, \alpha x_3) = \lambda(x_0, x_1, x_2, x_3)$$

Here $\lambda = 1, \alpha^2$ or α. For $\lambda = 1 \to x_1 = x_2 = x_3 = 0$ and the cubic becomes $x_0^3 = 0 \to x_0 = 0$ but $(0,0,0,0)$ is not allowed. Similarly for $\lambda = \alpha^2$. For $\lambda = \alpha \to x_0 = x_1 = 0$ and the cubic $\longrightarrow$ $x_2^3 + x_3^3 = 0 \to x_2^3 = e^{(2k+1)i\pi} x_3^3 \to x_2/x_3 = e^{\frac{(2k+1)i\pi}{3}}$ and we have 3 isolated fixed points in $\Sigma_1 \to \sum_j \chi(\mu_j) = 3$

$\longrightarrow \mathrm{Tr}_{H^{1,1}(\Sigma_1)}(g_1) = 1$

$\longrightarrow \mathrm{Tr}_{H^{1,1}(R_0)}(g) = 2$

the action of g on $H^{1,1}(R_0)$ can be represented as a 14×14 diagonal matrix with entries $1, \alpha, \alpha^2$ (g generates a $\mathbb{Z}_3$)

$\alpha + \alpha^2 = -1 \to$ in the above matrix we must have 6 ones, 4 α's and 4 α^2's $\to$ ($h^{1,1}(R) = 6$)

6 (1,1) harmonic forms are G-invariant
4 " " " transf. like $g : \psi \to \alpha \psi$
4 " " " " " $g : \psi \to \alpha^2 \psi$

$\overline{14}$

So from the $\overline{27}$ sector we have 6 leptons, 4 quarks, 4 antiquarks (all left-handed fields) we call them $\bar\lambda_1 \dots \bar\lambda_6$; $\bar Q_1, \dots, \bar Q_4$; $\bar q_1, \dots, \bar q_4$ ($g : \bar\lambda_i \to \bar\lambda_i$; $\bar Q_i \longrightarrow \alpha \bar Q_i$, $\bar q_i \to \alpha^2 \bar q_i$)

Discrete symmetries

The discrete symmetries of the CY-manifold R_0 depend on the particular choice of the defining polynomials, i.e., on the particular complex deformation of the manifold we choose.

Let us take the following choice:

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0$$
$$y_0^3 + y_1^3 + y_2^3 + y_3^3 = 0$$
$$x_0 y_0 + c_1 x_1 y_1 + c_2 (x_2 y_2 + x_3 y_3) = 0$$
$$c_1, c_2 \neq 1$$

We have the following discrete symmetries:

(i) $x_2 \leftrightarrow x_3$, $y_2 \leftrightarrow y_3$ (diagonal permutations in $(x_2 x_3)$ and $(y_2 y_3)$)

This is represented by $C = \left[\begin{array}{cc|cc} 1 & & & \\ & 1 & & \\ & & 0\ 1 & \\ & & 1\ 0 & \\ \hline & & 1 & \\ & & & 1 \\ & & & 0\ 1 \\ & & & 1\ 0 \end{array}\right]$ acting on $\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix}$.

(ii) $x_i \longrightarrow \alpha^{n_i} x_i$, $y_i \longrightarrow \alpha^{-n_i} y_i$

There is only one independent such transformation represented by

$$B = \left[\begin{array}{cccc|cccc} 1 & & & & & & & \\ & 1 & & & & & & \\ & & \alpha & & & & & \\ & & & 1 & & & & \\ \hline & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & \alpha^2 & \\ & & & & & & & 1 \end{array}\right]$$

All the other symmetries of this type can be generated thereof. Combining B and C we can get $x_4 \to \alpha x_4$, $y_4 \to \alpha^2 y_4$. The transformation

$$A = \left[\begin{array}{cccc|cccc} \alpha & & & & & & & \\ & \alpha & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ \hline & & & & \alpha^2 & & & \\ & & & & & \alpha^2 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{array}\right]$$

can also be generated by noting that an overall rescaling of x's and the inverse rescaling of y's (rescalings that are identities in the projective spaces) can bring this transformation to the x_2x_3, y_2y_3 subspace. The $x_0 \to \alpha x_0$, $y_0 \to \alpha^2 y_0$ can be generated by A,

$$g = \left[\begin{array}{cccc|cccc} 1 & & & & & & & \\ & \alpha^2 & & & & & & \\ & & \alpha & & & & & \\ & & & \alpha & & & & \\ \hline & & & & 1 & & & \\ & & & & & \alpha & & \\ & & & & & & \alpha^2 & \\ & & & & & & & \alpha^2 \end{array}\right] = \left[\begin{array}{c|c} g_1 & \\ \hline & g_2 \end{array}\right]$$

and transf's in the subspace $x_2x_3\ y_2y_3$.

(iii) $x_i \longleftrightarrow y_i$ (swapping operation)

which is represented by $D = \left[\begin{array}{cccc|cccc} & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \\ \hline 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \end{array}\right]$

The question is now which of these symmetries survive in $R = R_0/G$. In R the points x and $gx \in R_0$ are identified. So for a symmetry d to be defined on R, dx, dgx must also be identifiable

$\rightarrow \exists\, g' \in G \quad \ni \quad g' dx = dgx \longrightarrow$

$$d g d^{-1} = g'$$

It is obvious that B and C commute with g ($dgd^{-1} = g$) and therefore survive on R

The swapping

$$\left[\begin{array}{c|c} & \begin{smallmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{smallmatrix} \\ \hline \begin{smallmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{smallmatrix} & \end{array}\right] \left[\begin{array}{c|c} g_1 & \\ \hline & g_2 \end{array}\right] \left[\begin{array}{c|c} & \begin{smallmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{smallmatrix} \\ \hline \begin{smallmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{smallmatrix} & \end{array}\right] = \left[\begin{array}{c|c} g_2 & \\ \hline & g_1 \end{array}\right] = \left[\begin{array}{c|c} g_1^2 & \\ \hline & g_2^2 \end{array}\right]$$

$$(g_1^2 = g_2,\ g_2^2 = g_1)$$

So, for the swapping, we have $DgD^{-1} = g^2$ and the above condition is satisfied but in a non-trivial way $\longrightarrow$ swapping also survives on R

Thus the discrete symmetries on R are generated by B, C and D and there are no pseudosymmetries, i.e., symmetries of R_0 but not on R.

Now we will see what happens to the above symmetries by introducing flux breaking : $g \longrightarrow U_g \in E_6$

The action of d on a field $\psi(x)$ on R_0 can be defined as

$$d\,\psi(x) = \psi(dx)$$

But one can adopt a more general definition

$$d\,\psi(x) = V_d\, \psi(dx) \qquad \left(\frac{\text{the symmetry can be said to}}{\text{be} \quad d \otimes V_d}\right)$$

where $\quad d \longrightarrow V_d \in E_6$ is a homomorphism.

$$d\psi(gx) = U_g\, d\psi(x) = U_g V_d \psi(dx)$$
$$= V_d \psi(dgx) = V_d \psi(g'dx) = V_d U_{g'} \psi(dx)$$

$$\Longrightarrow \quad V_d^{-1} U_g V_d = U_{g'}$$ (A d survives flux breaking if a homo $d \to V_d$ exists so that this relation is satisfied)

In the trivial case $g' = g$, V_d can be chosen to be $V_d =$ identity
(for $d = B, C$)
For $d = D$, $g' = g^2$ and V_d cannot be chosen to be the identity

But in E_6 there is an element $\mathcal{J} : \underbrace{(A,B,C)}_{SU(3)_c \times SU(3)_L \times SU(3)_R} \longrightarrow (A^*, C^*, B^*)$

$$\to \quad \mathcal{J}^{-1} U_g \mathcal{J} = \mathcal{J}^{-1}(\alpha 1_3, 1_3, 1_3)\mathcal{J} = (\alpha^2 1_3, 1_3, 1_3) = U_g^2 = U_{g^2}$$

* So we can choose $V_D = \mathcal{J}$, and the swap operation survives flux breaking (we better say that $D \oplus \mathcal{J}$ survives flux breaking)

Thus, the group of "honest" symmetries is generated by B, C and $D \oplus \mathcal{J}$. There are no pseudosymmetries

B and C generate an 18-element non-abelian discrete group $\mathbb{V}$ which can be thought off consisting of all 2x2 diagonal or antidiagonal matrices (in the $x_2 x_3$-subspace) with entries $1, \alpha, \alpha^2 \longrightarrow$

$$\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}\alpha&0\\0&1\end{pmatrix}, \begin{pmatrix}\alpha^2&0\\0&1\end{pmatrix}, \begin{pmatrix}\alpha&0\\0&\alpha\end{pmatrix}, \begin{pmatrix}\alpha&0\\0&\alpha^2\end{pmatrix},$$
$$\begin{pmatrix}\alpha^2&0\\0&\alpha\end{pmatrix}, \begin{pmatrix}1&0\\0&\alpha\end{pmatrix}, \begin{pmatrix}1&0\\0&\alpha^2\end{pmatrix}, \begin{pmatrix}\alpha^2&0\\0&\alpha^2\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}, \begin{pmatrix}0&\alpha\\1&0\end{pmatrix}, \begin{pmatrix}0&\alpha^2\\1&0\end{pmatrix}$$
$$\begin{pmatrix}0&\alpha\\\alpha&0\end{pmatrix}, \begin{pmatrix}0&\alpha\\\alpha^2&0\end{pmatrix}, \begin{pmatrix}0&\alpha^2\\\alpha&0\end{pmatrix}, \begin{pmatrix}0&1\\\alpha&0\end{pmatrix}, \begin{pmatrix}0&1\\\alpha^2&0\end{pmatrix}, \begin{pmatrix}0&\alpha^2\\\alpha^2&0\end{pmatrix}.$$

$D \oplus \mathcal{J}$ generates a $\mathbb{Z}_2$-group which does not commute with $\mathbb{V}$

* The full group $\mathbb{W}$ has 36 elements of the form $\left[\begin{array}{c|c} f & \\ \hline & \tilde{f} \end{array}\right]$ and $\left[\begin{array}{c|c} & f \\ \hline \tilde{f} & \end{array}\right]$, where $f = \left(\begin{array}{c|c} \begin{smallmatrix}1&\\&1\end{smallmatrix} & \\ \hline & k \end{array}\right)$ with k one of 18 2x2 matrices and $\tilde{f} = C^{-m_1} B^{-l_1} C^{-m_2} \ldots$ for $f = C^{m_1} B^{l_1} C^{m_2} \ldots$

It is very easy to find how these transformations act on the λ's, q's and Q's from L-handed 27's. (One uses corresponding moments and applies B, C and $D \oplus \mathcal{J}$ on them) We then obtain

27's		A	B	C	[illegible]	[illegible]
Leptons	λ	α^2	α	$-\lambda$	$-\lambda$	1
	λ	α^2	1	$-\lambda$	$-\lambda$	1
	λ	α	α^2	$-\lambda$	$-\lambda$	1
	λ	α	1	$-\lambda$	$-\lambda$	1
	λ	1	1	$+1$	1	$-\lambda-\lambda$
	λ	1	1	$+1$	1	$-\lambda$
	λ	1	1	-1	1	1
	λ	1	α	$-\lambda$	$-\lambda$	1
	λ	1	α^2	$-\lambda$	$-\lambda$	1
Left handed quarks	q	α	α	$+1$	$-q$	
	q	α^2	α^2	$+1$	$-q$	
	q	1	1	$+1$	$-q$	
	q	α	α^2	$-q$	$-q$	$q-q$
	q	α	1	$-q$	$-q$	
	q	α^2	α	$-q$	$-q$	
	q	α^2	1	$-q$	$-q$	

Left	θ	α	α	$+1$	$-\theta$	
handed	θ	α^2	α^2	$+1$	$-\theta$	
anti-	θ	1	1	$+1$	$-\theta$	θ–θ
quarks	θ	α	α^2	$-\theta$	$-\theta$	
	θ	α	1	$-\theta$	$-\theta$	
	θ	α^2	α	$-\theta$	$-\theta$	
	θ	α^2	1	$-\theta$	$-\theta$	

Now we must find how the discrete symmetries act on the $\overline{27}$ sector.

We will restrict our selves for simplicity to the group $\mathbb{V}$ generated by B and C only (18-elements)

This group consists of 9 conjugacy classes ($g \sim hgh^{-1}, \forall h \in \mathbb{V}$)

(1) $\{1\}$

(2) $\{CBCB\}$

(3) $\{CB^2CB^2\}$

(4) $\{B, CBC\}$

(5) $\{B^2, CB^2C\}$

(6) $\{CB^2CB, CBCB^2\}$

(7) $\{C, BCB^2, B^2CB\}$

(8) $\{BCB, B^2C, CB^2\}$

(9) $\{BC, CB, B^2CB^2\}$

Then we know that $\mathbb{V}$ has 9 irreducible representations

The dimensionalities n_i $(i=1,2,\cdots,9)$ of these representations must satisfy

$$\sum_{i=1}^{9} n_i^2 = |\mathbb{V}| = \#\text{ of elements of } \mathbb{V} = 18$$

This means

$$n_1 = n_2 = \cdots = n_6 = 1 \quad , \quad n_7 = n_8 = n_9 = 2$$

The six one-dim. representations $R_1, \cdots, R_6$ can be generated by combining $B = 1, \alpha$ or α^2 with $C = 1$ or -1. The three 2-dim. R_7, R_8, R_9 are generated by $B = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \alpha^2 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^2 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

We can now form the table of characters of irreducible representations (character of $g \in \mathbb{V}$ in the representation R is $\mathrm{Tr}\,R(g)$; the characters are the same in a conjugacy class: $\mathrm{Tr}\,R(hgh^{-1}) = \mathrm{Tr}\,R(g)$):

Character table

← conjugacy classes

irreducible representations →

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
R1	1	1	1	1	1	1	1	1	1
R2	1	α^2	α	α	α^2	1	1	α^2	α
R3	1	α	α^2	α^2	α	1	1	α	α^2
R4	1	1	1	1	1	1	-1	-1	-1
R5	1	α^2	α	α	α^2	1	-1	$-\alpha^2$	$-\alpha$
R6	1	α	α^2	α^2	α	1	-1	$-\alpha$	$-\alpha^2$
R7	2	2α	$2\alpha^2$	$1+\alpha$	$1+\alpha^2$	-1	0	0	0
R8	2	$2\alpha^2$	2α	$1+\alpha^2$	$1+\alpha$	-1	0	0	0
R9	2	2	2	-1	-1	-1	0	0	0
$\overline{27}$ Leptons	[illegible]	0	0	0	0	[illegible]	[illegible]	[illegible]	[illegible]
$\overline{27}$ LH Quarks	[illegible]	[illegible]	4	[illegible]	[illegible]	-2	0	0	0
$\overline{27}$ LH Anti-Quarks	[illegible]	[illegible]	4	[illegible]	[illegible]	-2	0	0	0

For a group H with conjugacy classes H_k $(k = 1, \cdots, m)$ and irreducible representations R^λ $(\lambda = 1, \cdots, m)$ we can define χ_k^λ the character of H_k in the representation R^λ ($\chi_k^\lambda = \mathrm{Tr}\,R^\lambda(h_k)$, $h_k \in H_k$) ($|H_k| = \#$ of elements):

orthonormality between rows →
$$\sum_k \left(\frac{|H_k|}{|H|}\right)^{1/2} (\chi_k^\lambda)^* \left(\frac{|H_k|}{|H|}\right)^{1/2} \chi_k^\mu = \delta^{\lambda\mu}$$

orthonormality between columns →
$$\sum_\lambda \left(\frac{|H_k|}{|H|}\right)^{1/2} (\chi_k^\lambda)^* \left(\frac{|H_\ell|}{|H|}\right)^{1/2} \chi_\ell^\lambda = \delta_{k\ell}$$

These relations do hold for the above character table.

In order to find the transformation properties of the various fields of type $\overline{27}$ (remember they are harmonic (1,1)-forms) under $\mathbb{V}$ we must find the character of $\mathbb{V}$ acting on $H^{1,1}_{\bar\partial}(R_0)$ and then analyse it as a linear combination of characters of IR's $\longrightarrow$ this give the representation of $\mathbb{V}$ on $H^{1,1}$ as linear combination of IR's.

So, we must first calculate

$$\chi^{1,1}_{R_0}(d) = \mathrm{Tr}_{H^{1,1}(R_0)}(d) .$$

This can be done by Lefschetz fixed point theorem

It will be more usefull to split $H^{1,1}_{\bar\partial}(R_0)$ in three subspaces : leptons (6), quarks (4) and antiquarks (4).

The corresponding projection operator is

$$P_k = \frac{1}{3}\sum_{m=0}^{2} \alpha^{-km} g^m$$

which projects to leptons$(\bar\lambda_c)$, quarks$(\bar Q)$, antiquarks$(\bar q_i)$ for $k=0,1,2$.

Then we must calculate

$$\chi^{1,1}_{k}(d) = \frac{1}{3}\sum_{m=0}^{2} \alpha^{-km} \underbrace{\mathrm{Tr}_{H^{1,1}(R_0)}(g^m d)}_{\chi^{1,1}_{R_0}(g^m d)}$$

Lefschetz Hyperplane theorem $\longrightarrow$

$$\chi^{1,1}_{R_0}(f) = \chi^{1,1}_{\Sigma_1}(f) + \chi^{1,1}_{\Sigma_2}(f)$$

any

and $\chi^{1,1}_{\Sigma_1}(f)$ can be computed by Lefschetz fixed point theorem :

$$2 + \chi^{1,1}_{\Sigma_1}(f) = \sum_{P_j} \chi(P_j)$$

Let us have an example $f = (\alpha\ 1) = B$

Fixed points of B on the cubic Σ_1 :

$$B: (x_0 x_1 x_2 x_3) \longrightarrow (x_0 x_1 \alpha x_2 x_3) = \lambda (x_0 x_1 x_2 x_3)$$

$\longrightarrow \lambda = \alpha$ or 1 : (a) $\lambda = \alpha \rightarrow x_0 = x_1 = x_3 = 0$. Then the cubic $x_2^3 = 0$

$\rightarrow x_2 = 0 \rightarrow$ impossible

(b) $\lambda = 1 \rightarrow x_2 = 0$. Cubic $\rightarrow x_0^3 + x_1^3 + x_3^3 = 0$

The Euler characteristic of this set (a cubic in a $\mathbb{CP}^2$) is zero $\Big[\ c = \frac{(1+J)^{n+1}}{(1+d_1 J)\cdots(1+d_k J)}$, here we have k polyn. of degree $d_1,\dots,d_k$ in a $\mathbb{CP}^n$; J is the Kähler form of $\mathbb{CP}^n$ normalized so that $\int_{\mathbb{CP}^1} J = 1$ for any $\mathbb{CP}^1 \subset \mathbb{CP}^n \rightarrow \int_Y J^{n-k} = d_1 \cdots d_k$; the highest chern class is the Euler class. $\Big]$

$$\Rightarrow \sum_j \chi(P_j) = 0 \Rightarrow$$

$$\chi^{1,1}_{\Sigma_1}(B) = -2$$

For gB :

$$gB: (x_0 x_1 x_2 x_3) \longrightarrow (x_0, \alpha^2 x_1, \alpha^2 x_2, \alpha x_3) = \lambda (x_0 x_1 x_2 x_3)$$

$\longrightarrow$ Fixed points $x_0 = x_3 = 0$ $\quad x_1^3 + x_2^3 = 0 \longrightarrow$ three discrete points

$$\longrightarrow \sum_j \chi(P_j) = 3$$

$$\Rightarrow \quad \chi^{1,1}_{\Sigma_1}(gB) = 1$$

Similarly, $\quad \chi^{1,1}_{\Sigma_1}(g^2 B) = 1$

All this implies : $\quad \chi^{1,1}_{R_0}(B) = -4, \quad \chi^{1,1}_{R_0}(gB) = \chi^{1,1}_{R_0}(g^2 B) = 2$

$$\Rightarrow \quad \chi^{1,1}_0(B) = \frac{1}{3}\left(-4+2+2\right) = 0$$

$$\chi^{1,1}_1(B) = \frac{1}{3}\left(-4+2\alpha^2+2\alpha\right) = -2$$

$$\chi^{1,1}_2(B) = \frac{1}{3}\left(-4+2\alpha+2\alpha^2\right) = -2$$

Similarly, we can calculate the characters of all conjugacy classes of $\mathcal{V}$ in the lepton, quark, antiquark representations of $\overline{27}'$s

<u>The results are in the previous table</u>

We can then analyse these representations in IR's

$$* \left\| \begin{array}{l} \bar{\lambda} \to 2R_1 + 2R_9 + 2R_6 \\ \bar{Q} \to 2R_9 \\ \bar{q} \to 2R_9 \end{array} \right.$$

The transformation properties of the $\overline{27}'$s are then summarized in the following table:

$\overline{27}'$s		A	B	C		[illegible]	[illegible]
Leptons	[illegible]	1	1	+1	R_1	[illegible]	+1
	[illegible]	1	1	+1	R_1	[illegible]	+1
	[illegible]	α	α	−1	R_5	[illegible]	−1
	[illegible]	α	α	−1	R_9	[illegible]	−1
	[illegible]	α^2	α^2	−1	R_6	[illegible]	−1
	[illegible]	α^2	α^2	−1	R_6	[illegible]	−1
Left handed quarks	[illegible]	1	α	[illegible]	R_9	[illegible]	[illegible]
	[illegible]	1	α^2	[illegible]		[illegible]	
	[illegible]	1	α	[illegible]	R_9	[illegible]	
	[illegible]	1	α^2	[illegible]		[illegible]	
Left handed anti-quarks	[illegible]	1	α	[illegible]	R_9	[illegible]	[illegible]
	[illegible]	1	α^2	[illegible]		[illegible]	
	[illegible]	1	α	[illegible]	R_9	[illegible]	
	[illegible]	1	α^2	[illegible]		[illegible]	

By a similar analysis we can include $D \otimes \mathcal{Z}$ (the results are included in the above table.)

Remember the eqns that define our CY-manifold

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0$$

$$y_0^3 + y_1^3 + y_2^3 + y_3^3 = 0$$

$$x_0 y_0 + c_1 x_1 y_1 + c_2 (x_2 y_2 + x_3 y_3) = 0$$

(a) We can now take $\quad c_1 = 1 \;, \; c_2 \neq 1$

Then there exists an extra symmetry : $\begin{array}{l} x_0 \to x_1 \\ y_0 \to y_1 \end{array}$

$$P = \left[\begin{array}{cccc|cccc} 0 & 1 & & & & & & \\ 1 & 0 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ \hline & & & & 0 & 1 & & \\ & & & & 1 & 0 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{array}\right]$$

Since $PgP^{-1} = g^2$ (here we take into account that overall rescalings are unimportant)

$\rightarrow$ $P \oplus \mathbb{Z}$ is an "honest" symmetry

(b) we could even take $c_1 = c_2 = 1$, then we allow all diagonal permutations of $(x_0 x_1 x_2 x_3)$ and $(y_0 y_1 y_2 y_3)$.

For example

$$X = \left[\begin{array}{cccc|cccc} 1 & & & & & & & \\ & 0 & 1 & & & & & \\ & 1 & 0 & & & & & \\ & & & 1 & & & & \\ \hline & & & & 1 & & & \\ & & & & & 0 & 1 & \\ & & & & & 1 & 0 & \\ & & & & & & & 1 \end{array}\right], \ '''$$

These extra diagonal permutations do not become "honest" symmetries but remain as pseudosymmetries.

Pseudosymmetries (as well as 'honest' symmetries) give restrictions to Yukawa couplings.

This is due to the fact that Yukawa couplings on $R = R_0/G$ are computed by taking the Yukawa couplings on R_0 $(\lambda_{ijk}\phi^i\phi^j\phi^k)$ and by putting to zero the field components that do not transform properly on R $(\psi(gx) = U_g\psi(x))$ [Here ϕ_i's belong to E_6-representations and $\lambda_{ijk}\phi^i\phi^j\phi^k$ is an E_6 invariant coupling] This is correct since the starting Lagrangian does not change when $R_0 \rightarrow R = R_0/G$.

The above statement remains true to all orders because of the non-renormalization theorems of supersymmetric theories.

(because if they are R-sym. they do not apply on W directly)

A symmetry is called R-symmetry if it is not a symmetry of the superpotential W and $d^2\theta$ separately but only of the superspace integral $\int d^2\theta\, W$

$\rightarrow$ On CY-manifolds there is a covariantly const. spinor field η $(D_\mu\eta = 0)$

This spinor is related to the generator of four dim. supersymmetry.

W transforms like η^2

But there is an other quantity, namely $\varepsilon^{ijk} = \eta^T \Gamma^{ijk}\eta$ which transforms like η^2

Thus W transforms under global symmetries as the covariantly const. (3,0)-form ε^{ijk} does

$\Rightarrow$ For a symmetry not to be an R-symmetry, must leave $\varepsilon^{ijk} \in H^{3,0}_{\bar\partial}(R_0)$ invariant (remember ε^{ijk} is the form that generates the 1-dim. $H^{3,0}_{\bar\partial}(R_0)$)

We can now check if A is an R-symmetry or not (as an example)

$\Rightarrow$ We use Lefschetz fixed point theorem:

$$L(A) = \sum_{F_j} \chi(F_j) = \sum_{p,q=0}^{3} \mathrm{Tr}_{H^{p,q}_{\bar\partial}(R_0)}(A)$$

$$= 2\left[1 + \mathrm{Tr}_{H^{1,1}_{\bar\partial}(R_0)}(A) - \mathrm{Re}\left(\mathrm{Tr}_{H^{2,1}_{\bar\partial}(R_0)}(A)\right) - \mathrm{Re}(\xi_A)\right]$$

$(H^{3,0}, H^{0,3})$

Here, we used the Hodge diamond (R_0) and the fact that $\mathrm{Tr}_{H^{p,q}(R_0)}(A)$ and $\mathrm{Tr}_{H^{q,p}(R_0)}(A)$ are complex conjugate of each other.

Also ξ_A represents the action of A on ε^{ijk}:

$$A: \ \varepsilon \longrightarrow \xi_A\, \varepsilon$$

Since $P g P^{-1} = g^2$ (here we take into account that overall rescalings are unimportant)

$\longrightarrow$ $P \oplus J$ is an "honest" symmetry

(b) we could even take $c_1 = c_2 = 1$, then we allow all diagonal permutations of $(x_0 x_1 x_2 x_3)$ and $(y_0 y_1 y_2 y_3)$.

For example
$$X = \left[\begin{array}{cccc|cccc} 1 & & & & & & & \\ & 0 & 1 & & & & & \\ & 1 & 0 & & & & & \\ & & & 1 & & & & \\ \hline & & & & 1 & & & \\ & & & & & 0 & 1 & \\ & & & & & 1 & 0 & \\ & & & & & & & 1 \end{array}\right], \ldots$$

These extra diagonal permutations do not become "honest" symmetries but remain as pseudosymmetries.

Pseudosymmetries (as well as 'honest' symmetries) give restrictions to Yukawa couplings.

This is due to the fact that Yukawa couplings on $R = R_0/G$ are computed by taking the Yukawa couplings on R_0 ($\lambda_{ijk} \phi_i \phi_j \phi_k$) and by putting to zero the field components that do not transform properly on R ($\psi(gx) = U_g \psi(x)$) [Here ϕ_i's belong to E_6-representations and $\lambda_{ijk} \phi^i \phi^j \phi^k$ is an E_6 invariant coupling] This is correct since the starting Lagrangian does not change when $R_0 \to R = R_0/G$.

The above statement remains true to all orders because of the non-renormalization theorems of supersymmetric theories.

One should finally check whether our discrete symmetries are R-symmetries or not (because if they are R-sym. they do not apply on W directly)

A symmetry is called R-symmetry if it is not a symmetry of the superpotential W and $d^2\theta$ separately but only of the superspace integral $\int d^2\theta W$

$\to$ On CY-manifolds there is a covariantly const. spinor field η ($D_\mu \eta = 0$)

This spinor is related to the generator of four dim. supersymmetry.

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But there is an other quantity, namely $\epsilon^{ijk} = \eta^T \Gamma^{ijk} \eta$ which transforms like η^2

Thus W transforms under global symmetries as the covariantly const. (3,0)-form ϵ^{ijk} does

$\Rightarrow$ For a symmetry not to be an R-symmetry, must leave $\epsilon^{ijk} \in H^{3,0}_{\bar\partial}(R_0)$ invariant (remember ϵ^{ijk} is the form that generates the 1-dim. $H^{3,0}_{\bar\partial}(R_0)$)

We can now check if A is an R-symmetry or not (as an example)

$\Rightarrow$ We use Lefschetz fixed point theorem:

$$L(A) = \sum_{P_j} \gamma(P_j) = \sum_{p,q=0}^{3} \mathrm{Tr}_{H^{p,q}_{\bar\partial}(R_0)}(A)$$

$(H^{3,0}, H^{0,3})$

$$= 2\left[1 + \mathrm{Tr}_{H^{1,1}_{\bar\partial}(R_0)}(A) - \mathrm{Re}\left(\mathrm{Tr}_{H^{2,1}_{\bar\partial}(R_0)}(A)\right) - \mathrm{Re}(\xi_A)\right]$$

Here, we used the Hodge diamond (R_0) and the fact that $\mathrm{Tr}_{H^{p,q}(R_0)}(A)$ and $\mathrm{Tr}_{H^{q,p}(R_0)}(A)$ are complex conjugate of each other.

Also ξ_A represents the action of A on ϵ^{ijk}:

$$A: \ \epsilon \longrightarrow \xi_A \epsilon$$

From the known action of A on $27, \overline{27} \Rightarrow$

$$\mathrm{Tr}_{H^{1,1}(R_0)}(A) = 10 + 2(\alpha + \bar{\alpha}) = 10 - 2 = 8$$

$$\mathrm{Tr}_{H^{2,1}(R_0)}(A) = 7 + 8(\alpha + \bar{\alpha}) = -1 \quad \Rightarrow \mathrm{Re}\left(\mathrm{Tr}_{H^{2,1}(R_0)}(A)\right) = -1$$

Now the fixed points of A on R_0

$$(x_0 x_1 x_2 x_3) \to (\alpha x_0, \alpha x_1, x_2 x_3) = \lambda_1 (x_0 x_1 x_2 x_3)$$

$$(y_0 y_1 y_2 y_3) \to (\bar{\alpha} y_0, \bar{\alpha} y_1, y_2 y_3) = \lambda_2 (y_0 y_1 y_2 y_3)$$

$\to$ so, either (i) $x_0 = x_1 = 0$ or (ii) $x_2 = x_3 = 0$

" (a) $y_0 = y_1 = 0$ " (b) $y_2 = y_3 = 0$

(i) $\otimes$ (a) $\Rightarrow x_2^3 + x_3^3 = 0$, $y_2^3 + y_3^3 = 0$, $x_2 y_2 + x_3 y_3 = 0 \to x_2 \propto x_3$, $y_2 \propto y_3$

$x_3 y_3 = 0 \to$ impossible

(ii) $\otimes$ (b) $\Rightarrow$ (also) impossible

(i) $\otimes$ (b) $\Rightarrow x_2^3 + x_3^3 = 0$, $y_0^3 + y_1^3 = 0 \to (0,0,-\alpha^n,1) \otimes (1,-\alpha^m,0,0)$

(ii) $\otimes$ (a) $\Rightarrow$ similarly $\to (-\alpha^n,1,0,0) \otimes (0,0,1,-\alpha^m)$

$\to$ 18 isolated fixed points $\to \sum_{p_j} \chi(p_j) = 18$

LFPT $\to 18 = 2\left(1 + 8 - (-1) - \mathrm{Re}(\xi_A)\right)$

$\to \mathrm{Re}(\xi_A) = 1 \to \xi_A = 1$ ($|\xi_A| = 1$ since A is isometry)

$\Rightarrow$ A is not an R-symmetry

* One can check that none of the 'honest' and pseudo-symmetries discussed is an R-symmetry (They are all symmetries of W = superpotential and restrict Yukawa couplings as assumed)